Analytic solution of the separability criterion for continuous variable systems

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By using the algebraic separability criterion of R. Simon, we present an explicit determination of squeezing parameters for which the P-representation condition saturates the $Sp(2,R)\otimes Sp(2,R)$ invariant separability condition for continuous variable two-party Gaussian systems. We thus give for the first time the explicit analytic formulas of squeezing parameters which establish the equivalence of the separability condition with the P-representation condition. The implications of our algebraic analysis on some of the past related works are discussed.

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I. INTRODUCTION

The entanglement is a basic notion in quantum mechanics, but the quantitative criterion of entanglement is known only for a simple system such as a two-spin system [1, 2]. In view of this fact, it is remarkable that a proof of the necessary and sufficient separability criterion for continuous variable two-party Gaussian systems has been given by R. Simon [3] on the basis of generalized Peres-Horodecki criterion [1, 2]. He also gave an algebraic criterion for separability (*i.e.*, non-entanglement) [3], though it was not used in his proof. This algebraic criterion is considered to be fundamental, but its explicit analysis has not been performed so far. See a review [4] on the present status of the quantum separability problem of Gaussian systems.

We here present an explicit determination of squeezing parameters for which the P-representation condition saturates the $Sp(2,R)\otimes Sp(2,R)$ invariant separability condition by explicitly solving the algebraic condition of Simon. We thus give for the first time the explicit formulas of squeezing parameters, which establish the equivalence of the separability condition with the P-representation condition, in terms of the parameters of the standard form of the covariance matrix (or second moments) for Gaussian systems. These explicit analytic solutions should be useful in the quantitative theoretical and experimental analyses of entanglement such as in [5].

We also show that our analytic solutions of squeezing parameters r_1 and r_2 do not satisfy in general the equation $f(r_1^\star)=0$ which appears in another formulation of the separability criterion for two-party Gaussian systems [6]. In this sense our scheme is quantitatively different from the scheme in [6]. It is however shown that our exlicit analytic solutions allow us to construct a concrete proof of the separability criterion on the lines of [6].

II. ANALYTIC SOLUTIONS

We start with the 4×4 correlation matrix $V=(V_{\mu\nu})$ where

$$V_{\mu\nu}=\frac{1}{2}\langle\Delta\hat{\xi}_\mu\Delta\hat{\xi}_\nu+\Delta\hat{\xi}_\nu\Delta\hat{\xi}_\mu\rangle=\frac{1}{2}\langle\{\Delta\hat{\xi}_\mu,\Delta\hat{\xi}_\nu\}\rangle \tag{1}$$

with $\Delta\hat{\xi}_\mu=\hat{\xi}_\mu-\langle\hat{\xi}_\mu\rangle$ in term of the variables $(\hat{\xi}_\mu)=(\hat{q}_1,\hat{p}_1,\hat{q}_2,\hat{p}_2)$ for a two-party system specified by canonical variables $(\hat{q}_1,\hat{p}_1)$ and $(\hat{q}_2,\hat{p}_2)$. We generally define $\langle\hat{O}\rangle=\mathrm{Tr}\hat{\rho}\hat{O}$ by using the density matrix $\hat{\rho}$. The correlation matrix is also written in the form

$$V=\begin{pmatrix} A & C\\ C^T & B\end{pmatrix} \tag{2}$$

where A and B are 2×2 real symmetric matrices and C is a 2×2 real matrix. The standard form

$$V_0=\begin{pmatrix} a & 0 & c_1 & 0\\ 0 & a & 0 & c_2\\ c_1 & 0 & b & 0\\ 0 & c_2 & 0 & b\end{pmatrix} \tag{3}$$

is obtained from the general V by applying the $Sp(2,R)\otimes Sp(2,R)$ transformations [3].

The separability condition, which is derived from an analysis of the non-negativity of the partial transposed density matrix $\hat{\rho}$, is written in the matrix notation [3]

$$V+\frac{i}{2}\begin{pmatrix} J & 0\\ 0 & \pm J\end{pmatrix}\geq 0 \tag{4}$$

with a 2×2 simplectic matrix

$$J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \tag{5}$$

or equivalently

$$d^TAd+f^TBf+2d^TCf+g^TAg+h^TBh+2g^TCh \geq |d^TJg|+|f^TJh| \tag{6}$$

with $f\sim h$ all standing for *arbitrary* real two component vectors.

When one regards the separability condition as a constraint on the range of $|c_1|$ and $|c_2|$ in the standard form V_0 in (3), it is written as

$$4(ab-c_1^2)(ab-c_2^2) \geq (a^2+b^2) + 2|c_1c_2| - \frac{1}{4},$$
$$\sqrt{(2a-1)(2b-1)} \geq |c_1| + |c_2| \quad (7)$$

together with $a \geq 1/2$ and $b \geq 1/2$. The first algebraic relation in (7), which was derived by Simon [3], essentially corresponds to

$$4\det[V_0 + \frac{i}{2}\begin{pmatrix} J & 0 \\ 0 & \pm J \end{pmatrix}] \geq 0 \quad (8)$$

and thus it is manifestly invariant under $Sp(2,R) \otimes Sp(2,R)$. The second condition in (7) is given by the weaker conditions derived from (6) with *subsidiary* constraints $g = J^T d$ and $h = \pm J^T f$, and it is used to exclude the solutions of the first inequality in (7) which allow $c_1^2 \to \infty$ and $c_2^2 \to \infty$ for fixed a and b. The condition $a \geq \frac{1}{2}$, for example, is given by setting $g = J^T d$ and $h = f = 0$ in (6).

The separability condition (7) is explicitly solved, namely, the solution of the first inequality which satisfies the second constraint is given by

$$c_1^2 \;\leq\; \frac{1}{4t^2}\{[2ab(1+t^2)+t] - 2\sqrt{D(a,b,t)}\},$$
$$c_2^2 \;\leq\; \frac{1}{4}\{[2ab(1+t^2)+t] - 2\sqrt{D(a,b,t)}\} \quad (9)$$

with an auxiliary quantity

$$D(a,b,t) = a^2b^2(1-t^2)^2 + t(a+bt)(at+b) \quad (10)$$

for

$$0 \leq t \equiv |c_2|/|c_1| \leq 1 \quad (11)$$

where we choose $|c_2| \leq |c_1|$ without loss of generality. When one defines

$$f(c_1^2) = 4(ab-c_1^2)(ab-t^2c_1^2) - (a^2+b^2) - 2tc_1^2 + \frac{1}{4} \quad (12)$$

for the first inequality in (7), $f(c_1^2)$ assumes a negative minimum value at $c_1^2 = [2ab(1+t^2)+t]/(4t^2) > 0$ and $f(0) = 4(a^2-\frac{1}{4})(b^2-\frac{1}{4}) \geq 0$. Thus $f(c_1^2) = 0$ has two non-negative solutions. Since the second constraint in (7) is written as $(2a-1)(2b-1)/(1+t)^2 \geq c_1^2$, we examine

$$\begin{aligned} & f\left((2a-1)(2b-1)/(1+t)^2\right) \\ &= 4x^2(a-\frac{1}{2})^2(b-\frac{1}{2})^2 + 4(a^2-\frac{1}{4})(b^2-\frac{1}{4}) \\ & -4(a-\frac{1}{2})(b-\frac{1}{2})[ab(4-2x)+\frac{1}{2}x] \end{aligned} \quad (13)$$

with $0 \leq x = 4t/(t+1)^2 \leq 1$, which is confirmed to be negative for $0 \leq x < 1$ (for the generic case $a > \frac{1}{2}$ and $b > \frac{1}{2}$) and vanish for $x = 1, i.e., t = 1$. For $t = 1$ the solution in (9) coincides with the second constraint in (7), and thus only the solution in (9) satisfies the second constraint in (7). Note that $|c_1| = |c_2| = 0$ always defines a separable state.

Another essential ingredient in the proof of Simon [3] is the P-representation for Gaussian systems. The Gaussian state is called P-representable if the density matrix is written as (we assume $\langle\hat{\xi}_\mu\rangle = 0$ without loss of generality)

$$\hat{\rho} \;=\; \int d^2\alpha \int d^2\beta P(\alpha,\beta)|\alpha,\beta\rangle\langle\alpha,\beta| \quad (14)$$

where $|\alpha,\beta\rangle$ is the coherent state defined by $|\alpha,\beta\rangle = e^{\alpha\hat{a}^\dagger - \frac{1}{2}|\alpha|^2}|0\rangle \otimes e^{\beta\hat{b}^\dagger - \frac{1}{2}|\beta|^2}|0\rangle$ with $\hat{a} = (\hat{q}_1 + i\hat{p}_1)/\sqrt{2}$ and $\hat{b} = (\hat{q}_2 + i\hat{p}_2)/\sqrt{2}$. Thus the P-representable states are separable. By defining $\alpha = (\alpha_1 + i\alpha_2)/\sqrt{2}$ and $\beta = (\beta_1 + i\beta_2)/\sqrt{2}$, the weight factor $P(\alpha,\beta)$ in (14) is written in terms of the correlation matrix V as

$$\begin{aligned} & P(\alpha,\beta) \\ &= \frac{\sqrt{\det P}}{4\pi^2}\exp\{-\frac{1}{2}(\alpha_1,\alpha_2,\beta_1,\beta_2)P(\alpha_1,\alpha_2,\beta_1,\beta_2)^T\} \end{aligned} \quad (15)$$

with the matrix P given by

$$P^{-1} = V - \frac{1}{2}I \geq 0 \quad (16)$$

which defines the condition for the P-representation. See also [3].

The P-representation condition (16) is not invariant under $Sp(2,R) \otimes Sp(2,R)$ since $SS^T \neq I$ in general for $S \in Sp(2,R) \otimes Sp(2,R)$. We thus examine the allowed range of $Sp(2,R) \otimes Sp(2,R)$ transformations which preserve the condition $S^{-1}V_0(S^T)^{-1} - \frac{1}{2}I \geq 0$ or equivalently $V_0 - \frac{1}{2}SS^T \geq 0$ by starting with the standard form in (3). We choose the $Sp(2,R) \otimes Sp(2,R)$ squeezing matrix S as

$$S(r_1,r_2)S^T(r_1,r_2) = \begin{pmatrix} 1/r_1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & 1/r_2 & 0 \\ 0 & 0 & 0 & r_2 \end{pmatrix} \quad (17)$$

with suitably chosen $r_1 \geq 1$ and $r_2 \geq 1$. The eigenvalues of $V_0 - \frac{1}{2}S(r_1,r_2)S^T(r_1,r_2)$ are given by

$$\begin{aligned} (\lambda_1)_\pm \;&=\; \frac{1}{2}\{(a-\frac{1}{2r_1}) + (b-\frac{1}{2r_2}) \\ &\quad \pm\sqrt{((a-\frac{1}{2r_1}) - (b-\frac{1}{2r_2}))^2 + 4c_1^2}\}, \\ (\lambda_2)_\pm \;&=\; \frac{1}{2}\{(a-\frac{1}{2}r_1) + (b-\frac{1}{2}r_2) \\ &\quad \pm\sqrt{((a-\frac{1}{2}r_1) - (b-\frac{1}{2}r_2))^2 + 4c_2^2}\}. \end{aligned} \quad (18)$$

The P-representation exists if $(\lambda_1)_\pm \geq 0$ and $(\lambda_2)_\pm \geq 0$, namely, if the following two conditions are simultaneously

satisfied

$$(a-\frac{1}{2r_1})(b-\frac{1}{2r_2})\geq c_1^2,$$
$$(a-\frac{1}{2}r_1)(b-\frac{1}{2}r_2)\geq c_2^2 \quad (19)$$

together with $(a-\frac{1}{2r_1})+(b-\frac{1}{2r_2})\geq 0$ and $(a-\frac{1}{2}r_1)+(b-\frac{1}{2}r_2)\geq 0$.

When one regards (9) and (19) as constraints on the pair of variables (c_1^2, c_2^2) for given a and b, the P-representation condition is more restrictive than the separability condition; since the P-representation satisfies separablility, the set of points (c_1^2, c_2^2) allowed by the P-representation condition (19) always satisfy the separability condition (9). To be precise, we are working on the line defined by $t^2 = c_2^2/c_1^2$. See also (30) below. We thus expect that these two conditions can coincide only for the extremal value of the P-representation condition (19) with respect to r_1 and r_2 with fixed t.

We thus want to prove

$$\begin{aligned}&(a-\frac{1}{2r_1})(b-\frac{1}{2r_2(t,r_1)})\\&=\frac{1}{t^2}[(a-\frac{1}{2}r_1)(b-\frac{1}{2}r_2(t,r_1))]\\&=\frac{1}{4t^2}\{[2ab(1+t^2)+t]-2\sqrt{D(a,b,t)}\}\end{aligned} \quad (20)$$

for a suitable $1\leq r_1\leq 2a$ (and $1\leq r_2\leq 2b$) for any given $0\leq t\leq 1$ by regarding r_2 as a function of r_1 and t. By this way we establish that the separability condition (9) agrees with the P-representation condition (19) with a suitable $Sp(2,R)\otimes Sp(2,R)$ transformation.

We first consider the stationary points (or extremals) of $(a-\frac{1}{2r_1})(b-\frac{1}{2r_2(t,r_1)})$ and $(a-\frac{1}{2}r_1)(b-\frac{1}{2}r_2(t,r_1))$ in (19) with respect to r_1 with fixed t, namely

$$(b-\frac{1}{2r_2(t,r_1)})\frac{1}{r_1^2}+(a-\frac{1}{2r_1})\frac{1}{r_2^2}\frac{\partial r_2}{\partial r_1}=0,$$
$$(b-\frac{1}{2}r_2(t,r_1))+(a-\frac{1}{2}r_1)\frac{\partial r_2}{\partial r_1}=0. \quad (21)$$

These two relations combined give rise to

$$\frac{(ar_1-1/2)}{(a/r_1-1/2)}=\frac{(br_2-1/2)}{(b/r_2-1/2)}. \quad (22)$$

The relation (22) combined with the first equality in (20) gives

$$r_1=\frac{\frac{r_2}{t}a+\frac{1}{2}}{a+\frac{1}{2}\frac{r_2}{t}},\quad r_2=\frac{\frac{r_1}{t}b+\frac{1}{2}}{b+\frac{1}{2}\frac{r_1}{t}} \quad (23)$$

which are symmetric in r_1 and r_2 and are solved as

$$\begin{aligned}r_1&=\frac{1}{at+b}\{ab(1-t^2)+\sqrt{D(a,b,t)}\},\\ r_2&=\frac{1}{a+bt}\{ab(1-t^2)+\sqrt{D(a,b,t)}\}\end{aligned} \quad (24)$$

with $0\leq t=|c_2|/|c_1|\leq 1$ and $D(a,b,t)$ defined in (10). The squeezing parameters are thus determined.

One can confirm

$$2a\geq r_1\geq 1,\quad 2b\geq r_2\geq 1 \quad (25)$$

for $a\geq\frac{1}{2}$ and $b\geq\frac{1}{2}$, if one combines the relations (23) with $\infty > r_1/t\geq 1$ and $\infty > r_2/t\geq 1$ valid for the solutions in (24). The relation $\infty > r_1/t\geq 1$, for example, is established if one uses the expression

$$\frac{r_1}{t}=\frac{a+bt}{-ab(1-t^2)+\sqrt{a^2b^2(1-t^2)^2+t(a+bt)(at+b)}} \quad (26)$$

together with $t(at+b)\leq(a+bt)$ for $0\leq t\leq 1$ and the triangle inequality. Eq.(24) gives $r_1=r_2=1$ for $t=1$, and $r_1=2a,\ r_2=2b$ for $t=0$.

We emphasize that the condition (25) is required by a part of the P-representation condition $(a-\frac{1}{2r_1})+(b-\frac{1}{2r_2})\geq 0$ and $(a-\frac{1}{2}r_1)+(b-\frac{1}{2}r_2)\geq 0$ in (19) when combined with (22) which implies $r_2(1)=1$ and $r_2(2a)=2b$ if one regards $r_2(r_1)$ as a function of r_1. It should be noted that the squeezing parameters themselves are essentially determined by the first equality in (20) which comes from the P-representation condition.

We finally evaluate by using r_1 and r_2 in (24) as

$$\begin{aligned}&\frac{1}{t^2}(a-\frac{1}{2}r_1)(b-\frac{1}{2}r_2)\\&=\frac{1}{4t^2}\{[2ab(1+t^2)+t]-2\sqrt{D(a,b,t)}\}\end{aligned} \quad (27)$$

which is a remarkable identity. This relation establishes (20), namely, the fact that the boundaries of the conditions for separability and P-representation coincide for any $0\leq t=|c_2|/|c_1|\leq 1$. It is significant that the squeezing parameters in (24) depend only on the ratio $t=|c_2|/|c_1|$ and not on $|c_1|$ and $|c_2|$ separately unlike the case in [6]; this is because we are determining the bound to $|c_1|$ for given t to be consistent with (9). All the states parameterized by $|c_1|$, which satisfy (9) for given a, b and t, automatically satisfy the P-representation with the parameters in (24).

Our explicit construction proves that the P-representation condition (19) with the squeezing parameters in (24), which satisfy $1\leq r_1\leq 2a$ and $1\leq r_2\leq 2b$, is equivalent to the separability condition (9) for any $0\leq t=|c_2|/|c_1|\leq 1$, and thus the separability condition (9) is a necessary and sufficient separability criterion for two-party Gaussian systems. To our knowledge, our study is the first quantitative treatment of the basic algebraic condition (7) which is considered to be fundamental [4]. The existence of explicit analytic solutions of r_1 and r_2 in (24) for this basic problem is interesting, and the triplet

$$(V_0, r_1, r_2) \quad (28)$$

where V_0 is the standard form in (3), characterize the general covariance matrix (2) of either separable or inseparable Gaussian states. These explicit solutions are convenient for quantitative treatment, and thus they should

be useful for the quantitative theoretical and experimental analyses of entanglement in practical applications.

III. IMPLICATIONS OF ANALYTIC SOLUTIONS

We would like to compare the present analysis with some of the past related works. The separability condition (4) is based on the partial transpose operation of the density matrix in the manner of Peres and Horodecki[1, 2]. The analysis of the negativity of the partial transposed density matrix has been further extended by Shchukin and Vogel [7, 8]. We here note a complementary property that the second moment for the separable density matrix $\rho = \sum_n p_n \psi_n \psi_n^\dagger$ with $\psi_n = \phi_n(q_1)\varphi_n(q_2)$ gives rise to a generalization of (6)

$$\begin{aligned} &d^T A d + f^T B f + 2d^T C f + g^T A g + h^T B h + 2g^T C h \\ &\geq d^T \tilde{A} d + f^T \tilde{B} f + 2d^T \tilde{C} f + g^T \tilde{A} g + h^T \tilde{B} h + 2g^T \tilde{C} h \\ &+|d^T J g| + |f^T J h| \end{aligned} \quad (29)$$

by an analysis similar to the derivation of the Kennard's uncertainty relation *without* referring to the partial transpose operation. Here $\tilde{V}$ in terms of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ is defined analogously to (2) with the elements of the matrix $\tilde{V} = (\tilde{V}_{\mu\nu})$ defined by $\tilde{V}_{\mu\nu} = \sum_k p_k \langle \Delta\hat{\xi}_\mu \rangle_k \langle \Delta\hat{\xi}_\nu \rangle_k$ and $\Delta\hat{\xi}_\mu = \hat{\xi}_\mu - \langle \hat{\xi}_\mu \rangle$. Note that one may choose $\langle \hat{\xi}_\mu \rangle = \sum_k p_k \langle \hat{\xi}_\mu \rangle_k = 0$, but the average $\langle \hat{\xi}_\mu \rangle_k$ for each component state does not vanish in general. $\tilde{A}$ and $\tilde{B}$ are 2×2 real symmetric matrices and $\tilde{C}$ is a 2×2 real matrix. Both of V in (2) and $\tilde{V}$ are non-negative, and thus the condition (6) is necessary but not sufficient for separability in general. For the P-representation of the Gaussian state in (15) one can identify $P^{-1} = \tilde{V}$ by using a special property of the coherent state, which spoils $Sp(2, R) \otimes Sp(2, R)$ invariance due to normal ordering, and at the boundary of the P-representation condition where two eigenvalues of P^{-1} vanish, the condition (29) can be equivalent to (6). [9].

One can also directly derive the condition (6) from the P-representation condition (16) which implies

$$d^T A d + f^T B f + 2d^T C f - \frac{1}{2}(d^T d + f^T f) \geq 0 \quad (30)$$

for any d and f. When one adds (30) to a relation obtained from (30) by replacing d and f by g and h, respectively, one recovers the separability condition (6)

$$\begin{aligned} &d^T A d + f^T B f + 2d^T C f + g^T A g + h^T B h + 2g^T C h \\ &\geq \frac{1}{2}(d^T d + f^T f) + \frac{1}{2}(g^T g + h^T h) \geq |d^T J g| + |f^T J h|, \end{aligned}$$

where we used the relation such as $\frac{1}{2}(d^T d + g^T g) \geq \sqrt{(d^T d)(g^T J^T J g)} \geq |d^T J g|$.

What we have proved in the present paper (and also in [3]) is that (6) also implies (16) with the help of the squeezing operation. In this proof, the second weaker condition in (7) played a crucial role, namely, the first relation in (7) does not encode the entire information of the separability condition (4) or (6). The importance of the second condition in (7), which appears to be overlooked in the past related works (see, for example, eq.(19) in [3], Theorem V.2 in [4] and eq.(23) in [7]), was first recognized clearly in the present explicit analytic treatment of the separability condition (6).

As for the second weaker condition in (7), which is not $Sp(2, R) \otimes Sp(2, R)$ invariant, one can confirm that this weaker condition

$$\sqrt{[ar_1 + \frac{a}{r_1} - 1][br_2 + \frac{b}{r_2} - 1]} \geq \sqrt{r_1 r_2}|c_1| + \frac{|c_2|}{\sqrt{r_1 r_2}} \quad (31)$$

written for the second moment, which is obtained from the standard form V_0 in (3) by a squeezing operation S^{-1} in (17), corresponds to the separability condition in [6]; in fact, if one imposes the condition (22), the left-hand side of (31) becomes

$$\begin{aligned} &\sqrt{[ar_1 + \frac{a}{r_1} - 1][br_2 + \frac{b}{r_2} - 1]} \quad (32) \\ &= \sqrt{(ar_1 - 1/2)(br_2 - 1/2)} + \sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)} \end{aligned}$$

and one recovers eq. (16) in [6] when converted into their notation.

From our analysis of (13), it is obvious that some parameter region allowed by the weaker condition (31) with $r_1 = r_2 = 1$ does not satisfy the P-representation condition. But if one combines this weaker relation with suitable squeezing, a simple direct proof of the P-representation is obtained. If one sets $|c_2| = t|c_1|$ in (31) with the condition (22), one obtains

$$\begin{aligned} &\sqrt{(ar_1 - 1/2)(br_2 - 1/2)} + \sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)} \\ &\geq [\sqrt{r_1 r_2} + \frac{t}{\sqrt{r_1 r_2}}]|c_1|. \end{aligned} \quad (33)$$

If one uses the analytic formulas of squeezing parameters given in (24), one can confirm that the relation (33) when regarded as a bound to $|c_1|$ becomes identical to the $Sp(2, R) \otimes Sp(2, R)$ invariant bound (9) and also to the boundary of the P-representation condition (19). Here we use (20) and its square root, namely,

$$\begin{aligned} &\frac{\sqrt{(ar_1 - 1/2)(br_2 - 1/2)}}{\sqrt{r_1 r_2}} \\ &= \frac{\sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)}}{(t/\sqrt{r_1 r_2})} \\ &= \frac{1}{2t}\{[2ab(1 + t^2) + t] - 2\sqrt{D(a, b, t)}\}^{1/2}. \end{aligned} \quad (34)$$

Since (19) implies (33), one concludes that the weaker separability condition combined with our explicit squeezing parameters provides the necessary and sufficient condition for the P-representation. In a symbolic notation

we have

$$\text{eq.}(33) \supseteq \text{eq.}(9) \supseteq \text{eq.}(19), \tag{35}$$

which means, for example, any standard form of the covariance matrix V_0 which satisfies (19) automatically satisfies (9) (a stronger condition means a smaller set of V_0). But our analysis above shows that (33) and (19), both of which depend on squeezing parameters, coincide with (9) for our explicit solutions of squeezing parameters. This illustrates the power of our explicit analytic formulas, and this simple proof of the P-representation gives another explicit proof of the necessary and sufficient condition for separability of two-party Gaussian systems on the basis of (33).

We next briefly discuss the quantitative difference between our scheme and the scheme in [6]. The authors in [6] look for the solution $f(r_1^\star) = 0$ where

$$\begin{aligned} f(r_1) &= [\sqrt{r_1 r_2}|c_1| - \sqrt{(ar_1 - 1/2)(br_2 - 1/2)}] \quad (36) \\ &- [|c_2|/\sqrt{r_1 r_2} - \sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)}] \end{aligned}$$

and the condition (22) when written in our notation. Namely, they look for the solution of

$$\begin{aligned} &\sqrt{(ar_1 - 1/2)(br_2 - 1/2)} - \sqrt{r_1 r_2}|c_1| \\ &= \sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)} - |c_2|/\sqrt{r_1 r_2} \quad (37) \end{aligned}$$

together with (22) and (25), though the condition (25) is not explicitly stated in [6]. In contrast, in our scheme we solve

$$\begin{aligned} &\frac{\sqrt{(ar_1 - 1/2)(br_2 - 1/2)}}{\sqrt{r_1 r_2}|c_1|} \\ &= \frac{\sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)}}{(|c_2|/\sqrt{r_1 r_2})} \end{aligned} \tag{38}$$

together with (22) and (25), as is seen from the first equality of (34). Since (38) implies

$$\begin{aligned} &\frac{\sqrt{(ar_1 - 1/2)(br_2 - 1/2)} - \sqrt{r_1 r_2}|c_1|}{\sqrt{r_1 r_2}|c_1|} \\ &= \frac{\sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)} - |c_2|/\sqrt{r_1 r_2}}{(|c_2|/\sqrt{r_1 r_2})}, \end{aligned} \tag{39}$$

the *common* solutions of (37) and (38) exist only for

$$\begin{aligned} &\sqrt{(ar_1 - 1/2)(br_2 - 1/2)} - \sqrt{r_1 r_2}|c_1| \quad (40) \\ &= \sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)} - |c_2|/\sqrt{r_1 r_2} = 0 \end{aligned}$$

or else for $\sqrt{r_1 r_2}|c_1| = |c_2|/\sqrt{r_1 r_2}$, namely,

$$r_1 r_2 = \frac{|c_2|}{|c_1|} = t. \tag{41}$$

The relation (40) together with our explicit analytic solutions of r_1 and r_2 shows that the values of $|c_1|$ and $|c_2|$ given by (40) correspond to the largest allowed values of $|c_1|$ and $|c_2|$ in the separability condition (9) if one recalls (34). As for (41), we can choose $0 \le t \le 1$ without loss of generality as in (11) and the squeezing parameters are bounded as in (25) by the P-representation condition when combined with (22). Thus the condition (41) is satisfied only at $t = 1$ and $r_1 = r_2 = 1$. The two conditions (37) and (38) thus coincide only for such a measure-zero subset of the standard form of separable covariance matrices $\{V_0\}$. This quantitative difference between the two schemes is interesting.

As we explained already, the squeezing parameters in our scheme specify the boundary of the P-representation condition for any given t, which agrees with the separability condition (9). On the other hand, the scheme in [6] specifies the squeezing parameters for each given covariance matrix separately such that the P-representation condition is ensured, and thus the specification of squeezing parameters is more local. Our analytic solutions depend only on the ratio $t = |c_2|/|c_1|$, while squeezing parameters in (37) generally depend on $|c_1|$ and $|c_2|$ separately. Nevertheless, we here show that our explicit analytic solutions allow a concrete proof of the separability criterion in the scheme of [6]. For this purpose, we extend (35) to a symbolic notation

$$\text{eq.}(33) \supseteq \text{eq.}(9) \supseteq \text{eq.}(19) \supseteq \{\text{eq.}(33) \cap \text{eq.}(37)\} \tag{42}$$

where the last relation means that any V_0 which satisfies (37) in addition to (33) automatically satisfies the P-representation condition (19), as is explicitly confirmed [6]. Thus $\{\text{eq.}(33) \cap \text{eq.}(37)\}$ provides a *sufficient* condition for the P-representation, but the converse is not obvious. The P-representation condition (19) implies (33) as is shown in [6], but it is not obvious if (19) implies (37). To be more precise, it is not obvious if (9) implies (37). In fact, the authors in [6] choose a rather general class of V_0 and prove the solution of $f(r_1^\star) = 0$ in the domain $1 \le r_1^\star < \infty$ which, however, does not satisfy a part of the P-representation condition (25), namely, $1 \le r_1^\star \le 2a$.

We now show that (9) implies (37) by using our analytic solutions. We first write (36) in the form

$$\begin{aligned} &f(r_1, |c_1|) = (\sqrt{r_1 r_2} - t/\sqrt{r_1 r_2})|c_1| \quad (43) \\ &- \sqrt{(ar_1 - 1/2)(br_2 - 1/2)} + \sqrt{(a/r_1 - 1/2)(b/r_2 - 1/2)} \end{aligned}$$

together with (22). We then have $f(r_1 = 1, |c_1|) > 0$ for $0 \le t < 1$ (the case $t = 1$ is trivial since then $r_1 = r_2 = 1$ is the solution). From the analysis of (40), we have $f(r_1, |c_1|) = 0$ for the largest allowed $|c_1|$ in (9) and for our analytic solutions. Since $\sqrt{r_1 r_2} \ge 1$, the function $f(r_1, |c_1|)$ is a linear increasing function of $|c_1|$ when we fix squeezing parameters at our analytic solutions. Here it is crucial that our analytic solutions (24) depend only on a, b and t. We thus conclude

$$f(r_1, |c_1|) \le 0 \tag{44}$$

for all values of $|c_1|$ in (9) when the squeezing parameters are fixed at our analytic solutions. Since our analytic solutions satisfy $1 \le r_1 \le 2a$ as in (25), the relation (44)

combined with $f(r_1 = 1, |c_1|) > 0$ shows that $f(r_1^\star) = 0$ has a solution in the interval $1 \leq r_1^\star \leq 2a$ for all values of $|c_1|$ in (9). Namely, (9) implies (37). This completes the proof of the necessary and sufficient separability condition in the scheme of [6].

In the above analysis, we used the solution of (22)

$$r_2(r_1) = \frac{4b}{[\sqrt{(1-X)^2 + 16b^2X} + (1-X)]} \tag{45}$$

with $X(r_1) = (2a/r_1 - 1)/(2ar_1 - 1)$ which assumes $X(1) = 1$ and $X(2a) = 0$, and thus $r_2(1) = 1$ and $r_2(2a) = 2b$.

IV. DISCUSSION

We found the explicit analytic formulas of squeezing parameters which establish the equivalence of the $Sp(2,R) \otimes Sp(2,R)$ invariant separability condition with the P-representation condition. These explicit analytic formulas give rise to not only the concrete proofs of the separability criterions for two-party Gaussian systems formulated in [3] and [6] but also a new simple proof of the separability criterion as is described in connection with (35). Our analytic formulas thus provide a unified basis for the analysis of separability in continuous variable two-party Gaussian systems.

We analyzed the two-party system with one freedom for each party. The system we analyzed may be more properly called a two-mode system since the analysis of the two-party system with more than one freedom in each party is more involved [10]. As for the non-negativity of the partially transposed density matrix, Vidal and Werner [11] analyzed the separability of Gaussian states by using logarithmic negativity, which is essentially the same as the actual analysis of Simon [3] and leads to (7) and (16), but no analytic formulas of squeezing parameters are given. See also the related references [12] - [19].

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